Old UNM Algebra Qualifier Problems

Based on Chapters 7-8 of Dummit–Foote

Fifteen-year collection: January 2012–August 2023

Scope. This packet uses the third edition of Dummit and Foote, *Abstract Algebra*, Chapters 7 and 8 of Dummit and Foote: rings and ideals, quotient rings, localizations, matrix and power-series rings, Euclidean domains, principal ideal domains, and unique factorization domains. Problems whose standard solutions require the later theory of polynomial rings, modules, linear algebra, or field extensions have been excluded.

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1. January 2012 Qualifier, Problem 4

Prove that $\mathbb{Z}[\sqrt{10}]$ is not a unique factorization domain.

2. January 2012 Qualifier, Problem 5

Let F be a field and

$$R = F[x, x^2y, x^3y^2, x^4y^3, \ldots]$$

be a subring of the polynomial ring $F[x, y]$. Prove that the field of fractions of R is the same as $F(x, y)$, the field of fractions of $F[x, y]$.

3. August 2012 Qualifier, Problem 4

Let R be a domain. Prove that if R contains a nonprincipal proper ideal, then there is an ideal J maximal with this property. Prove that any such J is a prime ideal. Conclude that if all prime ideals of R are principal, then R is a principal ideal domain.

4. January 2013 Qualifier, Problem 6

Let $I = (x^5, y^4z^6)$ be an ideal of the polynomial ring $k[x, y, z]$. Determine the nilradical of $k[x, y, z]/I$.

5. January 2014 Qualifier, Problem 7

Determine how many elements the following rings have, and determine whether or not they are fields:

(a) $\mathbb{Z}[X]/(3, X^2 + 2)$;

(b) $\mathbb{Z}[X]/(5, X^3 + X + 1)$.

6. August 2014 Qualifier, Problem 3

Let R be a commutative ring and let $r \in R$ be a nonnilpotent element; that is, there is no natural number n such that $r^n = 0$.

(a) Show that there is a prime ideal P such that $r \notin P$.

(b) Give an example of a ring R and a nonnilpotent element r which is contained in every maximal ideal $\mathfrak{m}$ of R.

7. January 2015 Qualifier, Problem 3

Let R be a principal ideal domain and let S be a multiplicatively closed subset of R. Show that $S^{-1}R$ is a principal ideal domain.

8. January 2015 Qualifier, Problem 7

Prove that the ideal $(2, 1+\sqrt{-5})$ in $\mathbb{Z}[\sqrt{-5}]$ is not principal.

9. January 2017 Qualifier, Problem 4

Let I and J be two ideals in a commutative ring R such that $I + J = R$. Prove that:

(a) $IJ = I \cap J$;

(b) $I^n + J^n = R$ for all n.

10. August 2017 Qualifier, Problem 4

Suppose R and S are rings, $I \subseteq R$ is an ideal, and $J \subseteq S$ is an ideal. Show that $I \times J$ is an ideal of $R \times S$. Moreover, show that every ideal of $R \times S$ is of this form.

11. January 2019 Qualifier, Problem 4

Let R be a domain and let X be an indeterminate.

(a) Determine the units of $R[[X]]$.

(b) Show that $R[[X]]$ is a domain.

12. August 2019 Qualifier, Problem 1

Show that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.

13. August 2022 Qualifier, Problem 3

Let $R = M_2(\mathbb{Z}/6\mathbb{Z})$ be the ring of 2×2 matrices with entries in $\mathbb{Z}/6\mathbb{Z}$. Determine all ideals of R.

14. January 2024 Qualifier, Problem 4

Prove that if R is a commutative ring, $f \in R$ is a nonzero element, and S is the multiplicative system

$$S = \{f^n : n \geq 0\},$$

then there is a ring isomorphism

$$S^{-1}R \cong R[x]/(fx - 1).$$

15. August 2024 Qualifier, Problem 3

Let R be a ring satisfying $x^3 = x$ for all $x \in R$.

(a) Prove that the characteristic of R divides 6.

(b) Give two examples of rings, each of different characteristic, which satisfy $x^3 = x$ for all $x \in R$.

(c) If the characteristic of R is 2, show that R is Boolean, that is, $x^2 = x$, and hence commutative.